

The Classification of Finite Groups

Aim : Classify all finite groups up to isomorphism.

Step 1 : Show that any finite group G can be "broken down" into simple pieces.

Step 2 : Classify those simple pieces.

Step 3 : Understand how those simple pieces can fit together.

G - finite group

Definition A composition series for G is a collection of subgroups $\{e\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{r-1} \triangleleft G_r = G$ called a "chain" of subgroups

such that

1/ $G_{i-1} \triangleleft G_i$ if $i \in \{1, \dots, r\}$

2/ G_i / G_{i-1} is simple (ie has no non-trivial normal subgroups)
e.g. $\mathbb{Z}/p\mathbb{Z}$ p prime

Theorem Any finite group G has a composition series.

Must exist as $|G| < \infty$

Proof Let $r \in \mathbb{N}$ be the largest number such that there exists a chain

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{r-1} \triangleleft G_r = G$$

where $G_{i-1} \triangleleft G_i$. If G is simple then $r = 1$.

Claim : G_i / G_{i-1} is simple

If not then $\exists M \triangleleft \frac{G_i}{G_{i-1}}$ a non-trivial subgroup (ie $M \neq \{e\}$ and $M \neq \frac{G_i}{G_{i-1}}$)

3rd Isomorphism Theorem $\Rightarrow \exists H \subset G_i$, a subgroup, such that $\frac{H}{G_{i-1}} = M$.

$$M \neq \frac{G_i}{G_{i-1}} \Rightarrow H \neq G_i$$

$$M = \{e\} \Rightarrow G_{i-1} \neq H$$

$$M \triangleleft \frac{G_i}{G_{i-1}} \Rightarrow H \triangleleft G_i$$

$$G_{i-1} \triangleleft G_i \Rightarrow G_{i-1} \triangleleft H$$

$$\Rightarrow \{e\} = G_0 \neq G_1 \neq \dots \neq G_{i-1} \neq H \neq G_i \neq \dots \neq G_r = G$$

This is a longer chain with desired property.

This contradicts the maximality of r . Hence

$\frac{G_i}{G_{i-1}}$ is simple $\forall i \in \{1, \dots, r\}$ □

Example $\{e\} \neq \{e, (123), (132)\} \neq \text{Sym}_3 = G$

$\frac{G_1}{G_0} \cong G_1 \cong (\mathbb{Z}/3\mathbb{Z}, +) \leftarrow$ simple because 3 prime

$\{e, (123), (132)\} \triangleleft \text{Sym}_3$ and $[\text{Sym}_3 / \{e, (123), (132)\}] = 2$

$\Rightarrow \frac{G_2}{G_1} \cong (\mathbb{Z}/2\mathbb{Z}, +) \leftarrow$ simple because 2 prime

Jordan - Holder Theorem Let G be a finite

group. Suppose we have 2 composition series for G

$$\{e\} = G_0 \neq G_1 \neq \dots \neq G_{r-1} \neq G_r = G$$

$$\{e\} = H_0 \neq H_1 \neq \dots \neq H_{s-1} \neq H_s = G$$

Then $r = s$ and the list of quotients

$$\{G_r/G_{r-1}, \dots, G_1/G_0\} \text{ and } \{H_s/H_{s-1}, \dots, H_1/H_0\},$$

after reordering, are pairwise isomorphic.

Example $\{ \{0\} \} \subsetneq \{ \{0\}, [23], [47] \} \subsetneq \mathbb{Z}/6\mathbb{Z} = G$

" " "
 $G_0 \quad G_1 \quad G_2$

$\{ \{0\} \} \subsetneq \{ \{0\}, [3] \} \subsetneq \mathbb{Z}/6\mathbb{Z} = G$

" " "
 $H_0 \quad H_1 \quad H_2$

$\left\{ \begin{array}{c} G_2/G_1, G_1/G_0 \\ \text{ss} \quad \text{ss} \end{array} \right\}$ and $\left\{ \begin{array}{c} H_2/H_1, H_1/H_0 \\ \text{ss} \quad \text{ss} \end{array} \right\}$
 $(\mathbb{Z}/2\mathbb{Z}, +) \quad (\mathbb{Z}/3\mathbb{Z}, +) \quad (\mathbb{Z}/3\mathbb{Z}, +) \quad (\mathbb{Z}/2\mathbb{Z}, +)$

Definition If G has composition series

$$\{e\} = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_r = G$$

we call the quotients $\{ G_1/G_0, \dots, G_r/G_{r-1} \}$ the simple components of G .

By Jordan-Hölder the simple components of G are well-defined up to isomorphism.

Facts

1/ $\mathbb{Z}/p^n\mathbb{Z}$ has simple components $\overbrace{\{ \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, \dots, \mathbb{Z}/p\mathbb{Z} \}}^{n \text{ copies}}$

2/ Simple components of $G_1 \times \dots \times G_r = \bigcup_{i=1}^r \{ \text{Simple components of } G_i \}$

1/ and 2/ \Rightarrow

G Abelian $|G| = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ has simple components

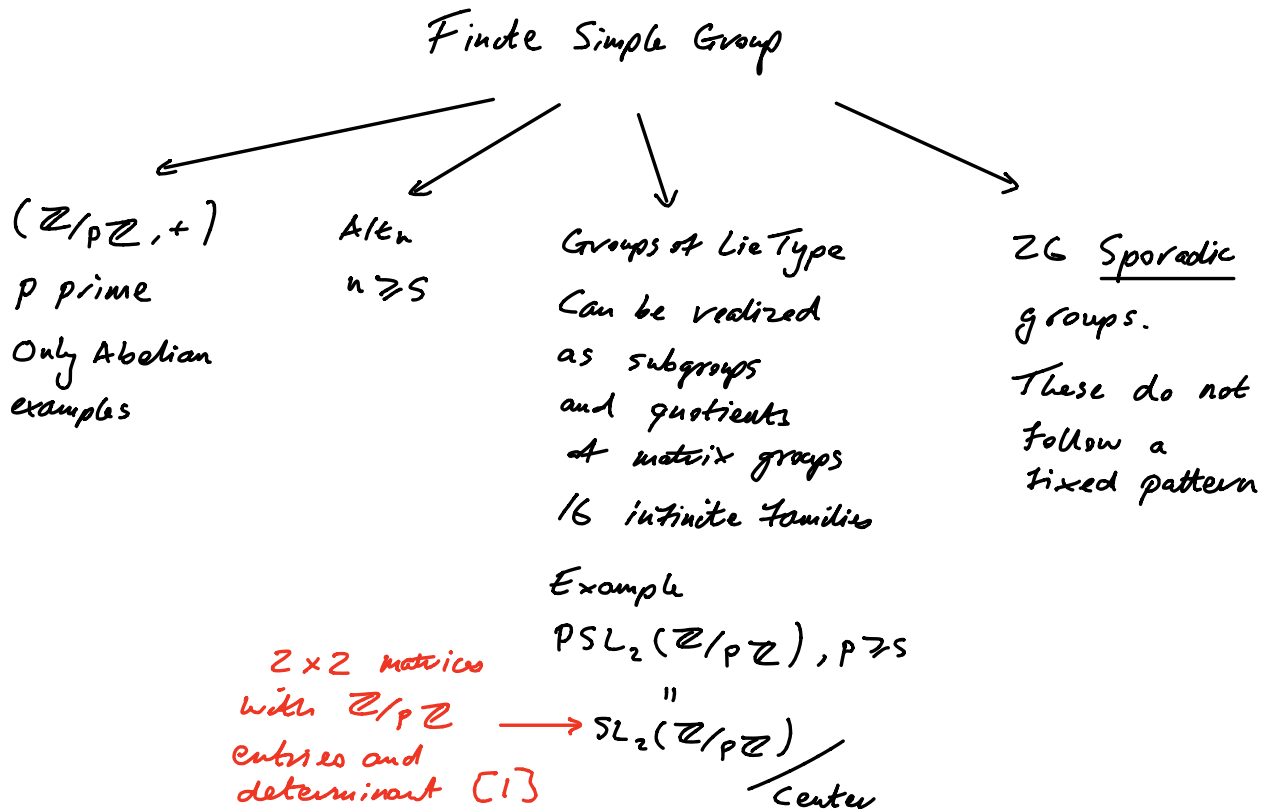
$\underbrace{\{ \mathbb{Z}/p_1\mathbb{Z}, \dots, \mathbb{Z}/p_1\mathbb{Z} \}}_{\alpha_1}, \underbrace{\{ \mathbb{Z}/p_2\mathbb{Z}, \dots, \mathbb{Z}/p_2\mathbb{Z} \}}_{\alpha_2}, \dots, \underbrace{\{ \mathbb{Z}/p_n\mathbb{Z}, \dots, \mathbb{Z}/p_n\mathbb{Z} \}}_{\alpha_n}$

WARNING: Two non-isomorphic groups can have the same simple components. For example $Sym_3, \mathbb{Z}/6\mathbb{Z}$ have $\{ \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \}$

Conclusion : Finite Group Theory is like Chemistry.

	Simple Groups	=	Atoms
Finite group with simple components $\{H_1, \dots, H_r\}$		=	Molecule with atoms $\{A_1, \dots, A_r\}$
Different finite groups with same simple components $\{H_1, \dots, H_r\}$		=	Different molecules with same atoms (Isomers) $\{A_1, \dots, A_r\}$
e.g. $Sym_3, \mathbb{Z}/6\mathbb{Z}$			e.g. silver cyanate and silver fulminate

Aim : Classify all finite simple groups up to isomorphism. (ie construct the periodic table of finite groups)



Move about Sporadic groups :

- Mathieu discovered the first

Subgroups with extremely strong transitivity properties

$$M_{11} \subset \text{Sym}_{11}, M_{12} \subset \text{Sym}_{12}, M_{22} \subset \text{Sym}_{22}, M_{23} \subset \text{Sym}_{23}, M_{24} \subset \text{Sym}_{24}.$$

- The largest is called the monster group.

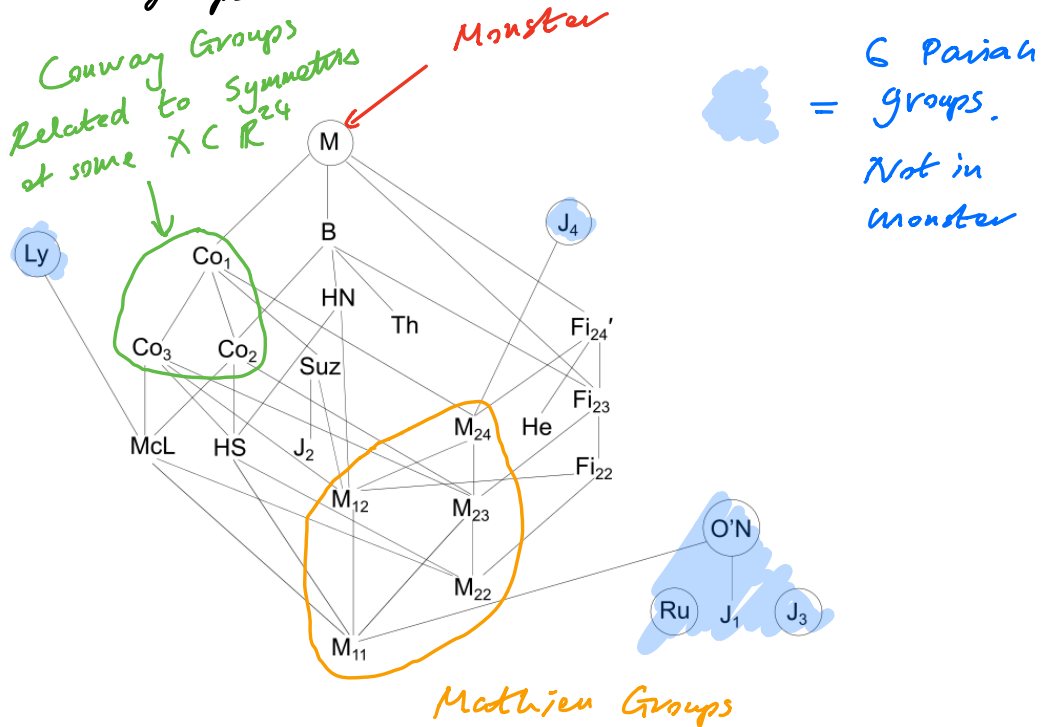
$$|\text{monster}| =$$

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

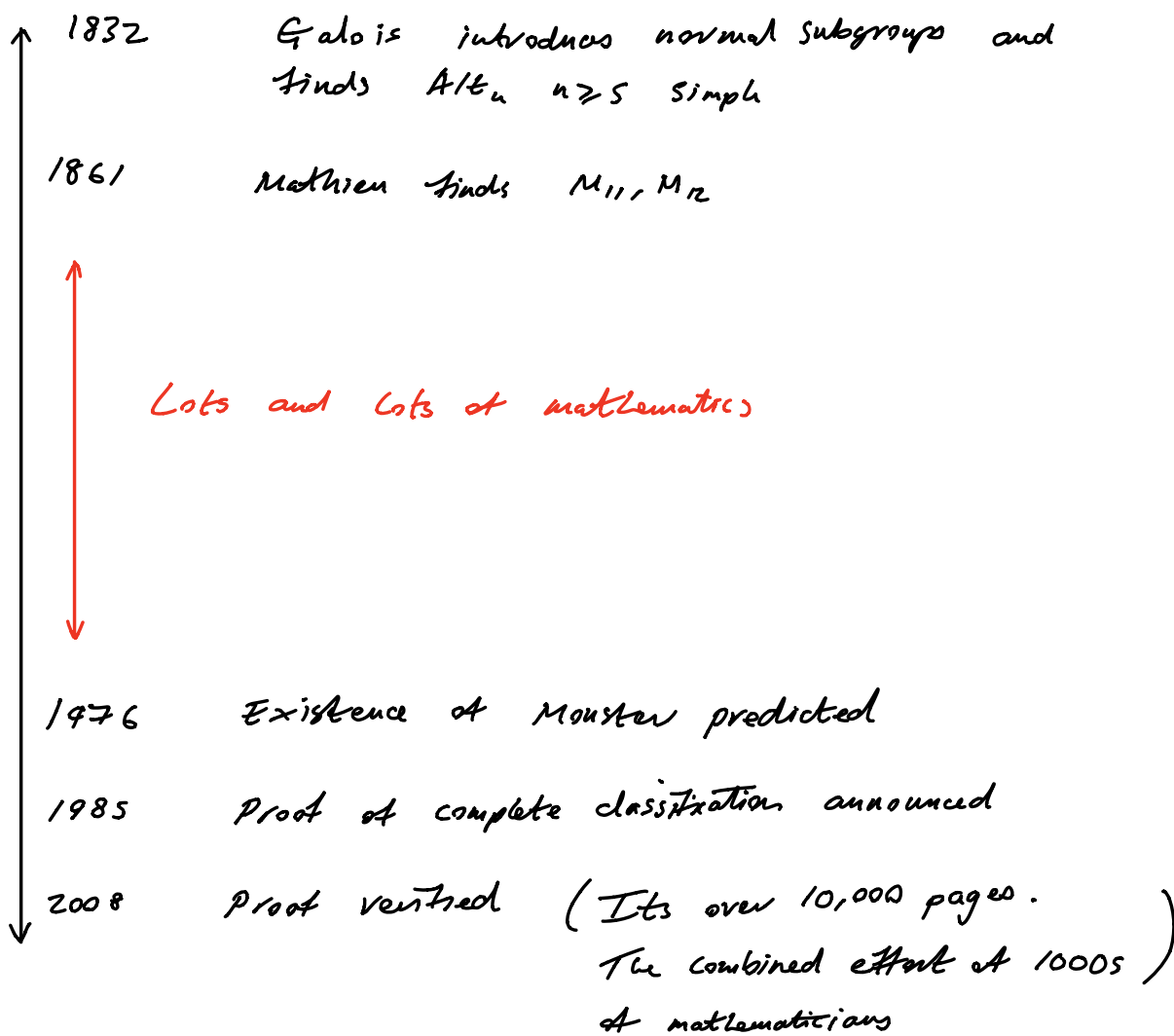
The monster turns up in strange places. For example

$$\text{monster} \cong \text{Subgroup of } GL_{196882 \times 196882}(\mathbb{Z}/2\mathbb{Z}).$$

The monster contains all but 6 sporadic groups as quotients of subgroups :



How did they prove all this?



The Periodic Table Of Finite Simple Groups

0, C ₂ , Z ₃		Dynkin Diagrams of Simple Lie Algebras															C ₂	
1																	2	
A ₁ (4), A ₁ (8)	A ₂ (2)																C ₃	
A ₅	A ₁ (7)																3	
60	168																C ₅	
A ₁ (9), B ₂ (2)'	² G ₂ (3)'																5	
A ₆	A ₁ (8)																C ₇	
360	504																7	
A ₇	A ₁ (11)	E ₆ (2)	E ₇ (2)	E ₈ (2)	F ₄ (2)	G ₂ (3)	³ D ₄ (2 ³)	² E ₆ (2 ²)	² B ₂ (2 ³)	Tits*	² F ₄ (2)'	² G ₂ (3 ³)	B ₃ (2)	C ₄ (3)	D ₅ (2)	² D ₄ (2 ²)	² A ₂ (9)	C ₂
2520	660	214 841 979 922 605 575 270 400	7 049 174 640 479 764 640 30 480 000 000	3 311 126 603 366 400	3 311 126 603 366 400	4 245 696 211 341 312	211 341 312 76 532 479 683 774 853 939 200	76 532 479 683 774 853 939 200	29 120	17 971 200	10 073 444 472	1 451 520	4 585 353 480	174 182 400	197 406 720	6 048		C ₃
A ₈ (2)	A ₁ (13)	E ₆ (3)	E ₇ (3)	E ₈ (3)	F ₄ (3)	G ₂ (4)	³ D ₄ (3 ³)	² E ₆ (3 ²)	² B ₂ (2 ⁵)	² F ₄ (2 ³)	² G ₂ (3 ⁵)	B ₂ (5)	C ₃ (7)	D ₄ (5)	² D ₄ (4 ²)	² A ₂ (16)		C ₅
20160	1 092	7 037 707 447 403 200 182 764 000 000	1 471 174 164 448 742 400 87 764 000 000	5 734 422 792 216 671 844 761 600	5 734 422 792 216 671 844 761 600	251 596 800	20 560 831 566 912	144 000 000 000 000 4 000 000 000 000	32 537 600	264 905 325 489 88 825 437	439 340 552	4 680 000	654 889 400	23 499 295 948 800	25 035 379 558 400	62 400		C ₇
A ₉	A ₁ (17)	E ₆ (4)	E ₇ (4)	E ₈ (4)	F ₄ (4)	G ₂ (5)	³ D ₄ (4 ³)	² E ₆ (4 ²)	² B ₂ (2 ⁷)	² F ₄ (2 ⁵)	² G ₂ (3 ⁷)	B ₂ (7)	C ₃ (9)	D ₅ (3)	² D ₄ (5 ²)	² A ₂ (25)		C ₁₁
181 440	2 448	10 526 756 796 344 000 182 764 000 000	1 471 174 164 448 742 400 87 764 000 000	5 734 422 792 216 671 844 761 600	5 734 422 792 216 671 844 761 600	5 839 000 000	67 802 350 642 790 400	4 000 000 000 000 4 000 000 000 000	34 093 383 680	239 389 910 264 332 349 332 432	138 297 600	54 025 731 402	274 457 216	8 913 509 400	17 880 203 200	67 536 471		C ₁₃
A _n	A _n (q)	E ₆ (q)	E ₇ (q)	E ₈ (q)	F ₄ (q)	G ₂ (q)	³ D ₄ (q ³)	² E ₆ (q ²)	² B ₂ (2 ²ⁿ⁺¹)	² F ₄ (2 ²ⁿ⁺¹)	² G ₂ (3 ²ⁿ⁺¹)	O _{2n+1} (q), O _{2n+1} (q')	PSp _{2n} (q)	O _{2n}^+(q)}	O _{2n}^-(q)}	PSU _{2n+1} (q)		Z _p
at 2	$\frac{q^n - 1}{q - 1}$	$\frac{q^6 - 1}{q - 1}$	$\frac{q^7 - 1}{q - 1}$	$\frac{q^8 - 1}{q - 1}$	$\frac{q^4 - 1}{q - 1}$	$\frac{q^2 - 1}{q - 1}$	$\frac{q^4 - 1}{q - 1}$	$\frac{q^2 - 1}{q - 1}$	$\frac{q^{2n+1} - 1}{q - 1}$	$\frac{q^{2n+1} - 1}{q - 1}$	$\frac{q^{2n+1} - 1}{q - 1}$	$\frac{q^{2n+1} - 1}{q - 1}$	$\frac{q^{2n} - 1}{q - 1}$	$\frac{q^{2n} - 1}{q - 1}$	$\frac{q^{2n} - 1}{q - 1}$	$\frac{q^{2n} - 1}{q - 1}$	$\frac{q^{2n} - 1}{q - 1}$	C _p
																		P

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates*
Symbol
Order [†]

M ₁₁	M ₁₂	M ₂₂	M ₂₃	M ₂₄	J(1), J(11)	HJ	HJM	J ₄	HS	McL	He	Ru
7920	95040	443 520	10 200 960	244 823 040	175 560	604 800	50 232 960	80 775 871 040 677 562 880	44 352 000	898 128 000	4 030 387 200	145 926 144 000

Sz	O'NS, O-S	-3	-2	-1	F ₄ , D	LyS	F ₄ , E	M(22)	M(23)	F ₃ , M(24)'	F ₂	F ₄ , M ₁
Suz	O'N	C ₀₃	C ₀₂	C ₀₁	HN	Ly	Th	Fi ₂₂	Fi ₂₃	Fi ₂₄ '	B	M
448 345 497 600	460 815 505 920	495 766 656 000	42 305 421 312 000	543 360 000	4 157 776 806 273 030	51 765 179 004 000 000	90 745 943 887 672 000	64 561 751 654 400	4 089 470 473 293 004 800	1 235 205 709 190 661 721 292 800	1 738 766 000 000 191 177 764 000 000	488 057 045 764 072 886 084 764 764 000 597 754 364 000 000

*The spin group $\Omega_n^{\epsilon}(2)$ is not a group of Lie type, but is the (index 2) commutator subgroup of $\Omega_n^{\epsilon}(2)$. It is usually given necessary Lie type status.

†The sporadic groups and families, alternate names in the upper left are other names by which they may be known. The specific non-sporadic groups are used to indicate isomorphisms. All such isomorphisms appear on the table except the family $\Omega_n^{\epsilon}(2^a) \cong C_n(2^a)$.

*Finite simple groups are determined by their order with the following exceptions:
 $B_2(q)$ and $C_2(q)$ for q odd $n > 2$.
 $A_n^{\epsilon}(2)$ and $A_1(4)$ of order 2048.

The groups starting on the second row are the classical groups. The sporadic simple group is unrelated to the families of finite groups.

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What's still to do? Chemistry didn't stop with the periodic table.

Open Problem: Classify all finite groups with given simple components.