The Classification of Finite Groups

Aim : Classify all finite groups up to isomorphism.

Step 1 : Show that any finite group G can be "broken down" into simple pieces.

Step 2 : Classify those simple pieces.

Step 3 : Understand how those simple pieces can fit together.

G - finite group

Definition A composition series for G is a collection of subgroups

called a "chain" of subgroups

$$\{e\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{r-1} \subsetneq G_r = G$$

such that

1. $G_{i-1} \triangleleft G_i \quad \forall\, i \in \{1, \ldots, r\}$
2. G_i / G_{i-1} is simple (ie has no non-trivial normal subgroups e.g. $\mathbb{Z}/p\mathbb{Z}$ p prime)

Theorem Any finite group G has a composition series.

Must exist as $|G| < \infty$

Proof Let $r \in \mathbb{N}$ be the largest number such that there exists a chain

$$\{e\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{r-1} \subsetneq G_r = G$$

where $G_{i-1} \triangleleft G_i$. If G is simple then $r = 1$.

Claim : G_i / G_{i-1} is simple

If not then $\exists\ M \lhd G_i/G_{i-1}$ a non-trivial subgroup (ie $M \neq \{eG_{i-1}\}$ and $M \neq G_i/G_{i-1}$)

3^{rd} Isomorphism Theorem $\Rightarrow \exists !\ H \subset G_i$, a subgroup, such that $H/G_{i-1} = M$.

$M \neq G_i/G_{i-1} \Rightarrow H \subsetneq G_i$

$M = \{eG_{i-1}\} \Rightarrow G_{i-1} \subsetneq H$

$M \lhd G_i/G_{i-1} \Rightarrow H \lhd G_i$

$G_{i-1} \lhd G_i \Rightarrow G_{i-1} \lhd H$

$\Rightarrow \{e\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{i-1} \subsetneq H \subsetneq G_i \cdots \subsetneq G_r = G$

This is a longer chain with desired property. This contradicts the maximality of r. Hence G_i/G_{i-1} is simple $\forall i \in \{1, \ldots, r\}$ $\square$

<u>Example</u>

$$\underset{\overset{\shortparallel}{G_0}}{\{e\}} \subsetneq \underset{\overset{\shortparallel}{G_1}}{\{e, (123), (132)\}} \subsetneq \underset{\overset{\shortparallel}{G_2}}{Sym_3} = G$$

$G_1/G_0 \cong G_1 \cong (\mathbb{Z}/3\mathbb{Z}, +) \leftarrow$ simple because 3 prime

$\{e, (123), (132)\} \lhd Sym_3$ and $\left| Sym_3/\{e,(123),(132)\} \right| = 2$

$\Rightarrow G_2/G_1 \cong (\mathbb{Z}/2\mathbb{Z}, +) \leftarrow$ simple because 2 prime

<u>Jordan – Holder Theorem</u> Let G be a finite group. Suppose we have 2 composition series for G

$$\{e\} = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_{r-1} \subsetneq G_r = G$$

$$\{e\} = H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_{s-1} \subsetneq H_s = G$$

Then $r = s$ and the list of quotients $\{G_r/G_{r-1}, \ldots, G_1/G_0\}$ and $\{H_s/H_{s-1}, \ldots, H_1/H_0\}$, after reordering, are pairwise isomorphic.

<u>Example</u> $\{[0]\} \subsetneq \{[0],[2],[4]\} \subsetneq \mathbb{Z}/6\mathbb{Z} = G$

$$\underset{G_0}{\{[0]\}} \subsetneq \underset{G_1}{\{[0],[2],[4]\}} \subsetneq \underset{G_2}{\mathbb{Z}/6\mathbb{Z}} = G$$

$$\underset{H_0}{\{[0]\}} \subsetneq \underset{H_1}{\{[0],[3]\}} \subsetneq \underset{H_2}{\mathbb{Z}/6\mathbb{Z}} = G$$

$$\left\{ \begin{matrix} G_2/G_1, & G_1/G_0 \\ \wr\wr & \wr\wr \\ (\mathbb{Z}/2\mathbb{Z},+) & (\mathbb{Z}/3\mathbb{Z},+) \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} H_2/H_1, & H_1/H_0 \\ \wr\wr & \wr\wr \\ (\mathbb{Z}/3\mathbb{Z},+) & (\mathbb{Z}/2\mathbb{Z},+) \end{matrix} \right\}$$

<u>Definition</u> If G has composition series

$$\{e\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$$

we call the quotients $\{G_1/G_0, \ldots, G_r/G_{r-1}\}$ the <u>simple components</u> of G.

By Jordan-Holder the simple components of G are well-defined up to isomorphism.

<u>Facts</u>

1) $\mathbb{Z}/p^n\mathbb{Z}$ has simple components $\{\underbrace{\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, \ldots, \mathbb{Z}/p\mathbb{Z}}_{n \text{ copies}}\}$

2) Simple components of $G_1 \times \cdots \times G_r$ $= \bigcup_{i=1}^{r} \{\text{Simple components of } G_i\}$

1) and 2) $\Rightarrow$

G Abelian $|G| = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ has simple compoents

$$\{\underbrace{\mathbb{Z}/p_1\mathbb{Z}, \ldots, \mathbb{Z}/p_1\mathbb{Z}}_{\alpha_1}, \underbrace{\mathbb{Z}/p_2\mathbb{Z}, \ldots, \mathbb{Z}/p_2\mathbb{Z}}_{\alpha_2}, \ldots, \underbrace{\mathbb{Z}/p_n\mathbb{Z}, \ldots, \mathbb{Z}/p_n\mathbb{Z}}_{\alpha_n}\}$$

WARNING: Two <u>non-isomorphic</u> groups can have the same simple components. For example Sym_3, $\mathbb{Z}/6\mathbb{Z}$ have $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$

<u>Conclusion</u> : Finite Group Theory is like Chemistry.

Simple Groups = Atoms

Finite group with simple components $\{H_1, \ldots, H_r\}$ = Molecule with atoms $\{A_1, \ldots, A_r\}$

Different finite groups with same simple components $\{H_1, \ldots, H_r\}$ = Different molecules with same atoms $\{A_1, \ldots, A_r\}$ (Isomers)

e.g. Sym_3, $\mathbb{Z}/6\mathbb{Z}$ — e.g. silver cyanate and silver fulminate

<u>Aim</u> : Classify all finite simple groups up to isomorphism. (ie construct the periodic table of finite groups)

Finite Simple Group

- $(\mathbb{Z}/p\mathbb{Z}, +)$, p prime. Only Abelian examples
- Alt_n, $n \geq 5$
- Groups of Lie Type. Can be realized as subgroups and quotients of matrix groups. 16 infinite families.

  Example: $PSL_2(\mathbb{Z}/p\mathbb{Z})$, $p \geq 5$

  $= SL_2(\mathbb{Z}/p\mathbb{Z}) / \text{Center}$

  2×2 matrices with $\mathbb{Z}/p\mathbb{Z}$ entries and determinant $\{1\}$ $\rightarrow SL_2(\mathbb{Z}/p\mathbb{Z})$
- 26 <u>Sporadic</u> groups. These do not follow a fixed pattern

More about Sporadic groups :

- Mathieu discovered the first

subgroups with extremely strong transitivity properties

$M_{11} \subset Sym_{11}$, $M_{12} \subset Sym_{12}$, $M_{22} \subset Sym_{22}$, $M_{23} \in Sym_{23}$, $M_{24} \subset Sym_{24}$.

- The largest is called the monster group.

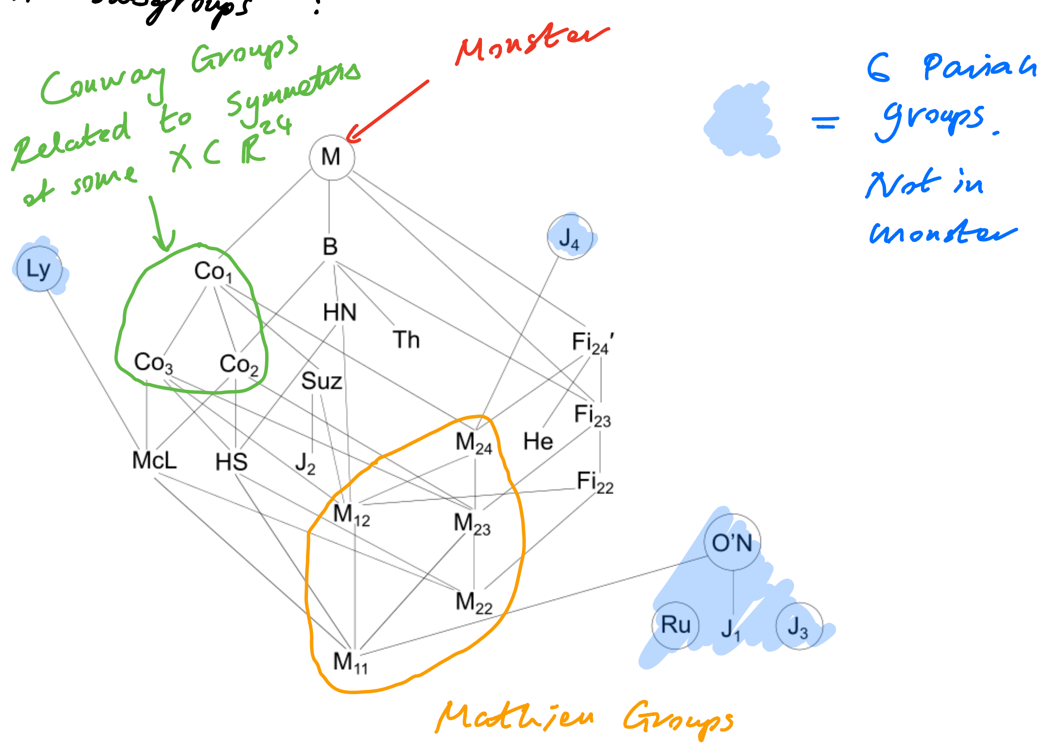
$|\text{monster}| =$

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

The monster turns up in strange places. For example

monster $\cong$ subgroup of $GL_{196882 \times 196882}(\mathbb{Z}/2\mathbb{Z})$.

The monster contains all but 6 sporadic groups as quotients of subgroups :



How did they prove all this?

1832 Galois introduces normal subgroups and finds Alt_n $n \geq 5$ simple

1861 Mathieu finds M_{11}, M_{12}

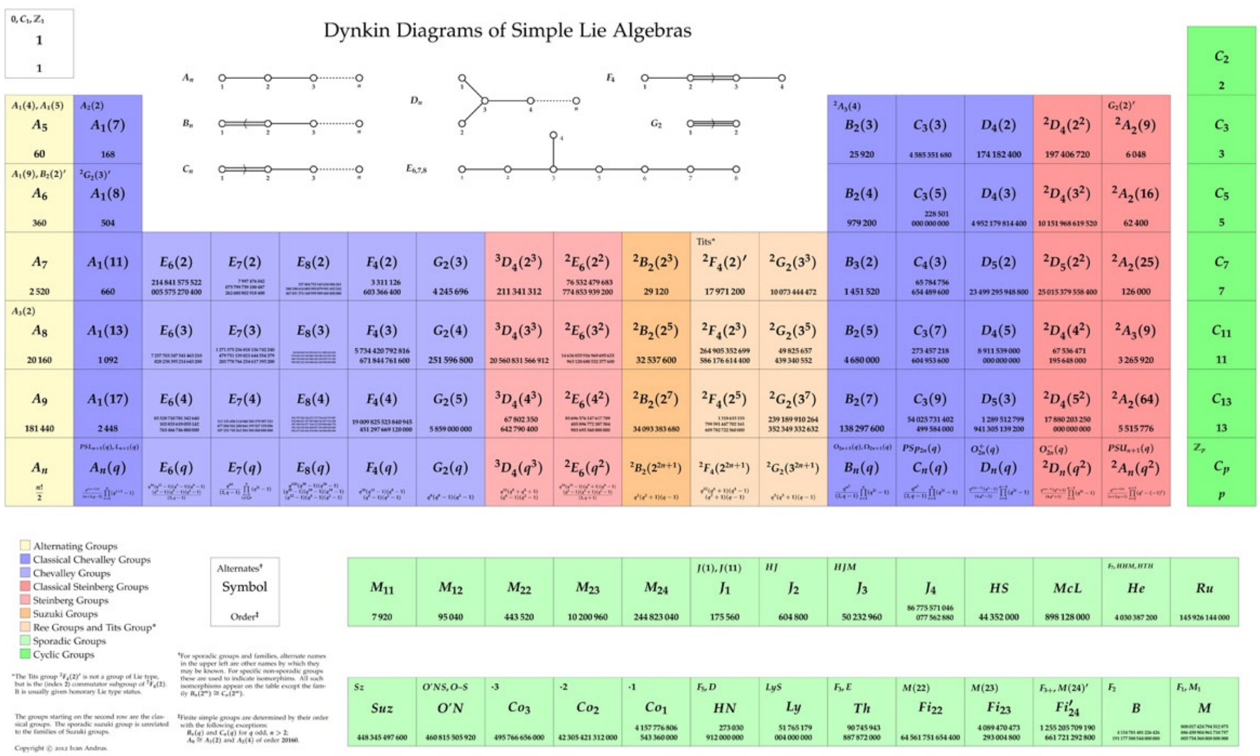
Lots and lots of mathematics

1976 Existence of Monster predicted

1985 Proof of complete classification announced

2008 Proof verified (Its over 10,000 pages. The combined effort of 1000s of mathematicians)

The Periodic Table Of Finite Simple Groups



What's still to do? Chemistry didn't stop with the periodic table.

<u>Open Problem</u> : Classify all finite groups with given simple components.